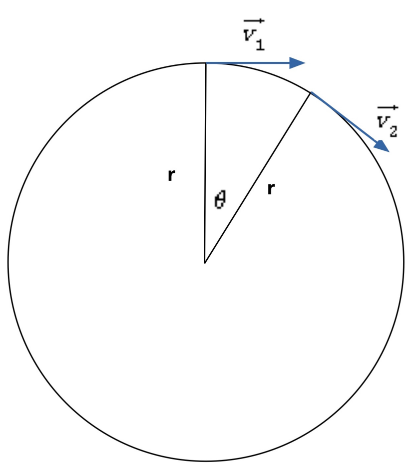
CENTRIPETAL FORCE

Circular Motion

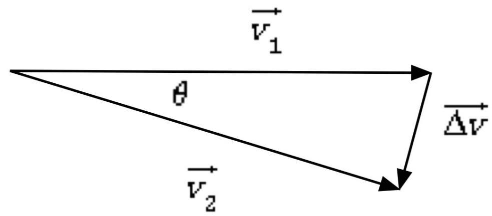
In the previous bulletin dealing with circular motion, we have already proven that all circular motion is accelerated motion. Now we will look a little further into that assertion.

ALL circular motion is accelerated motion.

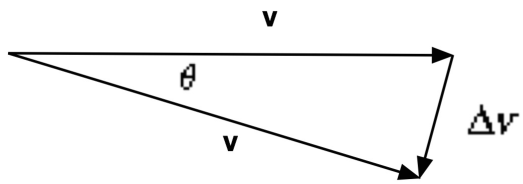
Let us consider the velocity of an object travelling in a circular motion such as a small object being spun at the end of a string. The object will have a velocity $\vec{v_1}$ at one point in the circle. The distance from this vector to the centre of the circle (the length of the string in our case) is the radius of the circle and denoted by the variable "r". Some time later, the object will have moved to some other position determined by the angle θ moved by the radius. The radius has not changed during this time interval.



If we were to add the two vectors, the difference between these two vectors would be $\overrightarrow{\Delta v}$, and $\vec{v_1} + \overrightarrow{\Delta v}$ would equal $\vec{v_2}$. See the following diagram.

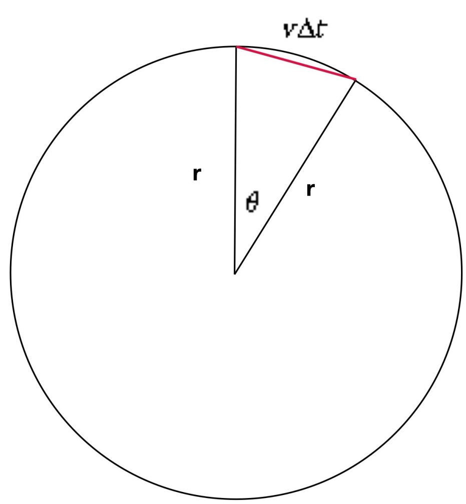


Note that the magnitude (length of the vectors) has not changed. The only thing that has changed is the direction of the vector. The speed of the object remains the same. Mathematically, we can denote this relationship as $v_1 = v_2 = v$ where "v" represents the speed of the object.

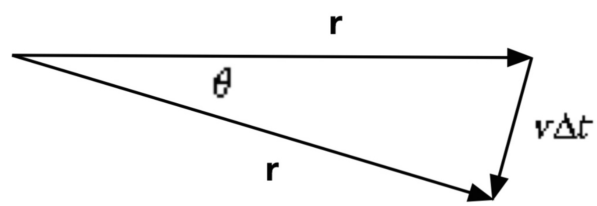


Geometrically this means that the two sides represented by the above triangle are of equal length. By definition, the above triangle is an isosceles triangle.

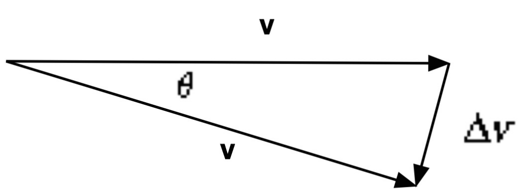
Now let us consider the path travelled by the object as the string moved a distance of θ degrees. The displacement between the starting and ending points covered by the object over the time interval Δt associated to the change in angle θ is equal to the speed times the time in motion or $v\Delta t$. This is approximated by the red line in the above diagram. Note that the displacement between these two points (red line) is nearly the same length as the distance travelled along the circumference of the circle.

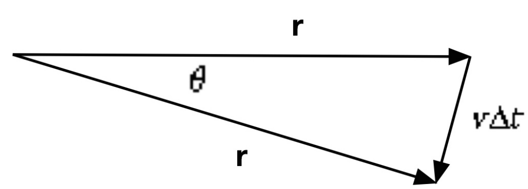


When the triangle within this circle is isolated, the following diagram is obtained.



Now let us consider the relationship between two of the triangles that we have developed.





Both are isosceles triangles; both have two sides of identical length. Also, both identical sides are separated by the same angle θ. Therefore, there is a ratio between all sides of both triangles. As a result, the following relationship exists between both triangles.

$$\frac{\Delta v}{v} \cong \frac{v\Delta t}{r}$$

Note: The “approximately equal” sign ($\cong$) was used. This is because the distance covered by $v\Delta t$ is an approximation to the distance covered by the circumference of the circle over the angle θ.

When Δv is isolated from the above equation, the following is obtained.

$$\Delta v \cong \frac{v^2 \Delta t}{r}$$

Since acceleration is defined as

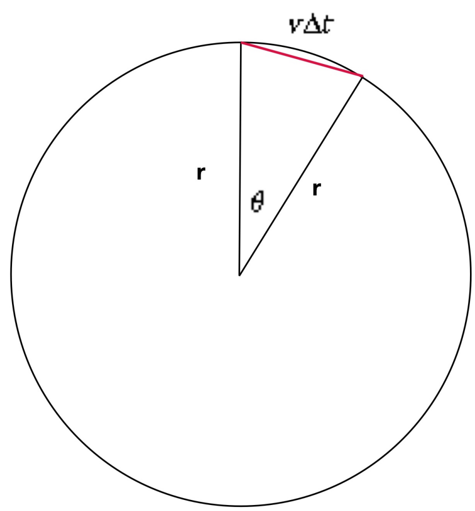
$$a = \frac{\Delta v}{\Delta t}$$

It follows that

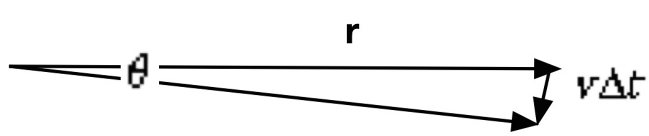
$$a \cong \frac{v^2}{r}$$

The next step is to reduce the error associated with the above equation. In essence, we want the "approximately equal" symbol ($\cong$) with the "equal" symbol (=).

Let us re-examine the following diagram.



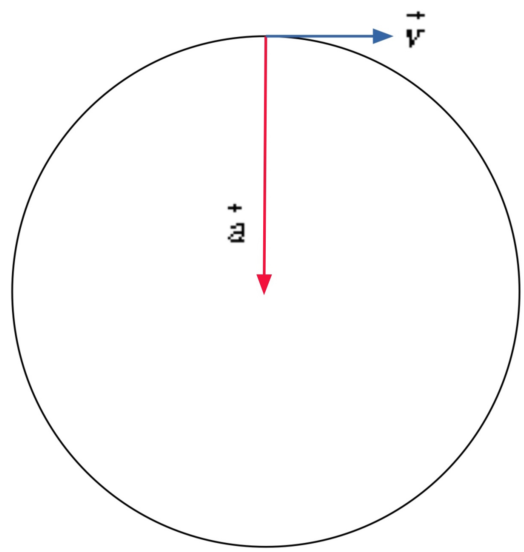
The displacement between these two points (red line) is nearly the same length as the distance travelled along the circumference of the circle. However, it is important to note that the difference between the distance and the displacement decreases as the angle θ decreases. As the angle θ approaches its limit of zero, the difference between the displacement and the distance approaches zero. Mathematically, this means that the error in the approximation used by the $v\Delta t$ approaches zero. At very small angles where θ approaches zero, the pie shape in the above diagram more closely resembles a line and the angle between the radius and $v\Delta t$ approached 90 degrees. This is the basis of calculus.



Under such conditions, the **instantaneous** acceleration can be determined at any point in the circular motion. The magnitude of the acceleration is

$$a = \frac{v^2}{r}$$

Also, the direction of the acceleration is always at right angles to the velocity vector and directed **inward** toward the centre of motion.



Since the force actin on an object is defined as the product of mass and its acceleration,

$$\vec{F} = m\vec{a}$$

and since

$\vec{a} = \frac{\overrightarrow{v^2}}{r}$ directed inwards toward the centre of motion,

It follows that

$\vec{F_c} = \frac{m\overrightarrow{v^2}}{r}$ at right angles to the velocity vector and inwards toward the centre of motion.

Centripetal force ($\overrightarrow{F_c}$) is defined as follows:

$$\overrightarrow{F_c} = \frac{m\overrightarrow{v^2}}{r}$$

and is directed at right angles to the velocity vector and toward inward toward the centre of motion.